Exercise 12

Evaluate the line integral, where C is the given curve.

$$\int_C (x^2 + y^2 + z^2)\, ds, \quad C:\ x = t,\ y = \cos 2t,\ z = \sin 2t, \quad 0 \le t \le 2\pi$$

Solution

With this parameterization in t, the line integral becomes

$$\begin{aligned}
\int_C (x^2 + y^2 + z^2)\, ds &= \int_0^{2\pi} \left\{[x(t)]^2 + [y(t)]^2 + [z(t)]^2\right\} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \\
&= \int_0^{2\pi} (t^2 + \cos^2 2t + \sin^2 2t)\sqrt{(1)^2 + (-2\sin 2t)^2 + (2\cos 2t)^2}\, dt \\
&= \int_0^{2\pi} (t^2 + 1)\sqrt{1 + 4\sin^2 2t + 4\cos^2 2t}\, dt \\
&= \int_0^{2\pi} (t^2 + 1)\sqrt{1 + 4}\, dt \\
&= \sqrt{5}\int_0^{2\pi} (t^2 + 1)\, dt \\
&= \sqrt{5}\left(\frac{t^3}{3} + t\right)\Big|_0^{2\pi} \\
&= \sqrt{5}\left(\frac{8\pi^3}{3} + 2\pi\right) \\
&= \frac{2\pi\sqrt{5}}{3}(4\pi^2 + 3).
\end{aligned}$$